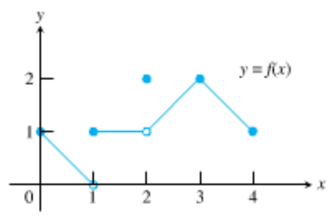
Exploration 2-4a

Continuity at a Point

Function f is continuous at $x = c$ if and only if

1. $f(c)$ exists,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Continuity at a Point



EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

DEFINITION **Continuity at a Point**

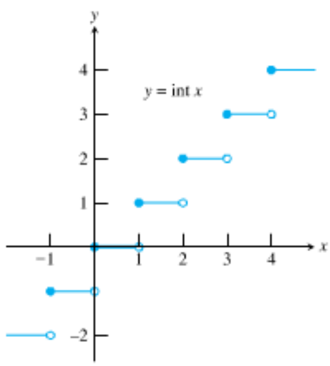
Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

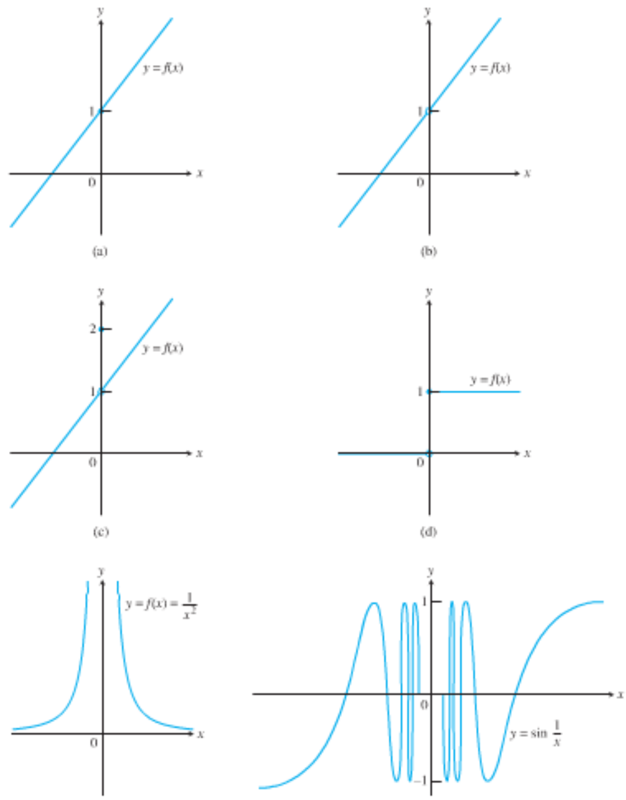


EXAMPLE 2 **Finding Points of Continuity and Discontinuity**

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x\to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** it oscillates and has no limit as $x \to 0$.



EXPLORATION 1 Removing a Discontinuity

Let $f(x) = \frac{x^3 - 7x - 6}{x^2 - 9}$.

$\frac{x^3-7x-6}{(x+3)(x-3)}$

1. Factor the denominator. What is the domain of f? $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f. (use synthetic division)

$\frac{(x-3)(x+2)(x+1)}{(x-3)(x+3)}$

$\lim_{x \to 3} f(x) = 10/3$

5. Show that the *extended function*

$$g(x) = \begin{cases} \frac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now try Exercise 25.

Continuity on an Interval

- Function f is continuous on an interval of x-values if and only if it is continuous at each value of x in that interval. At the endpoints of a closed interval, only the one-sided limits need to equal the function value.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

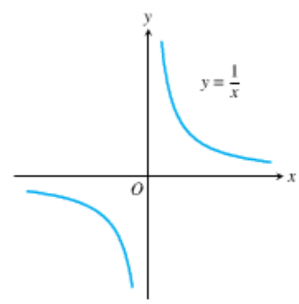


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \to c$ is $g(f(c))$.

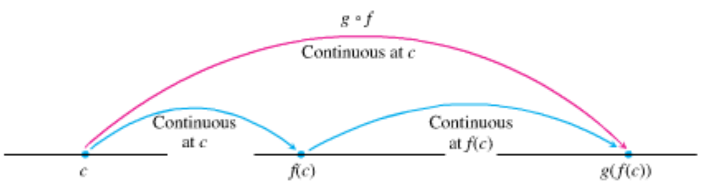


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

EXAMPLE 4 Using Theorem 7

Show that $y = \left|\frac{x \sin x}{x^2 + 2}\right|$ is continuous.

Exploration 2-6m

Exploration 2-6m: The Intermediate Value Theorem

Objective: To discover and acquire a feel for one of the major theorems in calculus.

1. Let $f(x)=x^5-4x^2+1$. Find $f(1)$ and $f(2)$.

$f(1)=-2$

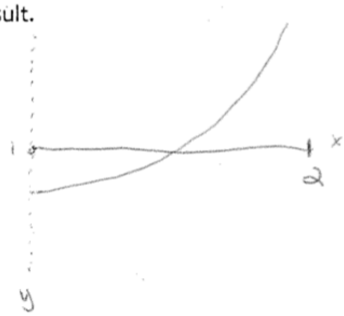
$f(2)=17$

2. Explain why you think the graph of f must cross the x-axis somewhere between $x=1$ and $x=2$.

there is a sign change

and f(x) is a continuous function

3. Plot the graph of $y=f(x)$ on the closed interval [1, 2]. Sketch the result.



4. Find the zeros (roots) of f on [1, 2].

$x=1.529$

5. Let

$$g(x)=\begin{cases} x^2+1 & \text{for } -3\le x\le 0 \\ 1-x^2-x^4 & \text{for } 0<x\le 2 \end{cases}$$

. Find $g(-3)$ and $g(2)$.

$g(-3)=10$

$g(2)=-19$

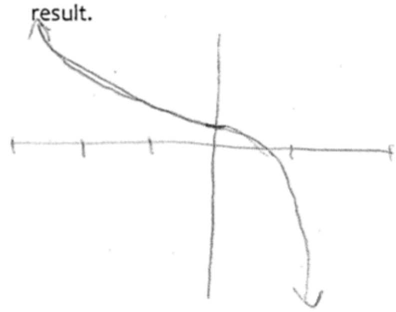
6. Explain why you think the graph of g must cross the x-axis somewhere between $x=-3$ and $x=2$.

$\lim_{x\to 0^-} g(x)=1$ $\quad g(0)=1$

$\lim_{x\to 0^+} g(x)=1$ $\quad \therefore$ continuous

sign changes

7. Plot the graph of $y=g(x)$ on the closed interval [-3, 2]. Sketch the result.



8. Find the zeros (roots) of g on [-3, 2].

$x=.786$

Exploration 2-6m: The Intermediate Value Theorem

Objective: To discover and acquire a feel for one of the major theorems in calculus.

$h(3) = 10$

$h(2) = -21$

9. Now replace g with the slightly altered function h defined on [-3, 2] by

$$h(x) = \begin{cases} x^2+1 & \text{for } -3 \le x \le 0 \\ -1-x^2-x^4 & \text{for } 0 < x \le 2 \end{cases}$$

Find $h(-3)$ and $h(2)$.

$\lim_{x\to 0^-} h(x) = 1$

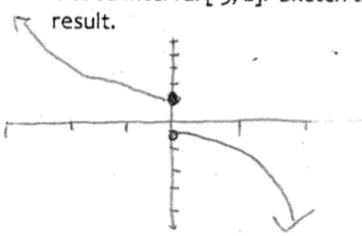
$\lim_{x\to 0^+} h(x) = -1$

not continuous

10. Do you think the graph of h must cross the x-axis somewhere between $x = -3$ and $x = 2$? Explain your answer.

not necessarily

11. Plot the graph of $y = h(x)$ on the closed interval [-3, 2]. Sketch the result.



12. Explain why the graph of g had to cross the x-axis in [-3, 2] but the graph of h does not.

this graph is not continuous at ~~x~~ =0

13. You are now ready to formulate a statement of the Intermediate Value Theorem. Based upon the observations above, fill in the blanks to complete the following.

Given a <u>Continous</u> function f defined on the closed interval $[a, b]$ for which 0 is between <u>$f(a)$</u> and <u>$f(b)$</u>, there exists a point c between <u>a</u> and <u>b</u> such that <u>$f(c)=0$</u>.

14. Let $g(x) = x^5 - 4x^2 + 1$. Find $g(1)$ and $g(2)$.

$g(1) = -2$

$g(2) = 17$

15. Let $f(x) = g(x) - 7$. Use your statement of the intermediate value theorem above to show that there must be some value between 1 and 2 such that f is equal to 0 at that x-value.

since ~~g~~ f(x) is continuous and 0 is between -9 and 10 there is some c in which ~~g~~ f(c)=0

... the x-axis

Exploration 2-6m: The Intermediate Value Theorem
Objective: To discover and acquire a feel for one of the major theorems in calculus.

16. Must g then be equal to 7 somewhere on [1, 2]? Explain your answer.

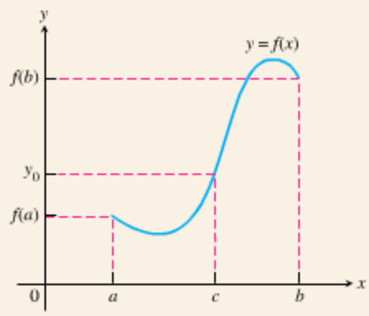
    yes, ~~it's continuous and must~~ be equal to because ~~f(x)~~ in order for f(x) to = 0 g(x) must be equal to 7

17. Generalize the ideas behind exercises 14 – 16 to complete, with understanding, the following statement.

General Intermediate Value Theorem: Suppose the function g is continuous on the closed interval [*a*, *b*]. For any real number *d* between f(a) and f(b) there exists a point *c* between a and b such that f(c)=d

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



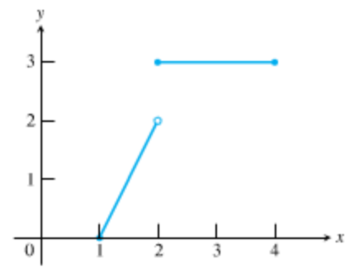


Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of sin x. It will not have jumps like those in the graph of the greatest integer function int x, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points *(dot mode)*. Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode)*. For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.